Variational sets: calculus and applications to optimality conditions in nonsmooth vector optimization

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Abstract. We develop elements of calculus of variational sets, which were recently introduced in [1, 2] to replace generalized derivatives in establishing optimality conditions in nonsmooth optimization. As these conditions are the major goal of considering generalized derivatives, we also discuss applications in obtaining higher-order optimality conditions in vector optimization. Among various kinds of optimality concepts we focus on the Benson-proper efficiency and Q-optimal solution, which are attracting remarkable attentions.

Keywords: Higher-order variational sets, calculus rules, higher-order optimality conditions, local Benson-proper solutions, local Q-minimal solution.

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1. Introduction

In nonsmooth optimization, many generalized derivatives have been introduced to replace the Fréchet and Gateaux derivatives which do not exist. Each of them is adequate for some classes of problems, but not all. A generalized derivative being effectively used or not depends on probably first how can one employ it to establish optimality conditions and second, whether it enjoys good properties and calculus rules. In [1, 2] we proposed two kinds of variational sets for mappings between normed spaces. These subsets of the image space are larger than the images of the pre-image space through known generalized set-valued mappings. Hence our necessary optimality conditions obtained by separation techniques are stronger than many known conditions using various generalized derivatives. Of course, sufficient optimality conditions based on separations of bigger sets may be weaker. But in [1, 2], using variational sets we can establish sufficient conditions which have almost no gap with the corresponding necessary ones. The second advantage of the variational sets is that we can define these sets of any order to get higher-order optimality conditions. This feature is significant since many important and powerful generalized derivatives can be defined only for the first and second orders and the higher-order optimality conditions available in the literature are much fewer than the first and second-order ones. The third strong point of the variational sets is that almost no assumptions are needed to be imposed for their being well-defined and nonempty and also for establishing optimality conditions. Calculating them from the definition is only a computation of the Kuratowski-Painlevé limit. However, in [1, 2] no calculus rules for variational sets are provided.

The aim of the present paper is to establish elements of calculus for variational sets and provide selected applications in optimality conditions. Most of the usual rules, from the sum and chain rules to various operations in analysis, are investigated. It turns out that the variational sets possess many fundamental and comprehensive calculus rules. Although this construction is not comparable with objects in the dual approach like Mordukhovich's coderivatives (see the excellent books [3, 4]) in enjoying rich calculus, it may be better in dealing with higher-order properties. As applications and illustrations we choose the Benson-proper [5] and Q-minimal solutions [6] as representatives for a wide range of solution concepts. Note that the Q-minimality unifies weak, ideal efficiencies as well as most of proper efficiency notions in vector optimization.

The organization of the paper is as follows. The rest of this section is devoted to recalling definitions needed in the sequel. We present the two kinds of higher-order variational sets, including various equivalent formulations and simple properties in Section 2. In the next Section 3 we explore comprehensive calculus rules for the variational sets. We also try to illustrate by examples the unfortunate lack of expected rules. We provide in Section 4 simple applications of the variational sets in establishing higher-order conditions for the local Benson-proper and local Q-minimal solutions to a nonsmooth set-valued vector optimization with general inequality constraints.

Throughout the paper, if not otherwise specified, let X and Y be real normed spaces, $C \subseteq Y$ a closed pointed convex cone with nonempty interior and $F : X \to 2^Y$. For $A \subseteq X$, int$A$, cl$A$ (or $\bar{A}$), bd$A$ denote its interior, closure and boundary, respectively. X^* is the dual space of X and B_X stands for the closed unit ball in X. For $x_0 \in X$, $U(x_0)$ is used for the set of all neighborhoods of $x_0 \in X$. $\mathbb{R}^k_+$ is the nonnegative orthant of the k-dimensional space. For $r \in \mathbb{R}$ tending to 0, $0(r)$ and $\vartheta(r)$ mean a moving point z in the space in question (always clear from the context) such that $\frac{1}{r}\|z\| \to 0$ and $\|z\| \to 0$, respectively. We often use the following cones, for $A \subseteq X$, C above and $u \in X$,

$$\text{cone}A = \{\lambda a \mid \lambda \geq 0,\ a \in A\},$$

$$\text{cone}_+ A = \{\lambda a \mid \lambda > 0,\ a \in A\},$$

$$A(u) = \text{cone}(A + u),$$

$$C^* = \{y^* \in Y^* \mid \langle y^*, c\rangle \geq 0,\ \forall c \in C\} \text{ (polar cone)},$$

$$C^\sharp = \{y^* \in Y^* \mid \langle y^*, c\rangle > 0,\ \forall c \in C \setminus \{0\}\} \text{ (quasi} - \text{interior of } C^*).$$

A nonempty convex subset B of a convex cone C is called a base of C if $C = \text{cone}B$ and $0 \notin \text{cl}B$. For a subset $A \subseteq X$, the contingent cone of A at $x_0 \in X$ is

$$T_A(x_0) = \{u \in X \mid \exists t_n \to 0^+, \exists u_n \to u,\ \forall n, x_0 + t_n u_n \in A\}.$$

For $H : X \to 2^Y$, the domain, graph and epigraph of H are defined as

$$\text{dom}H = \{x \in X : H(x) \neq \emptyset\},\ \text{gr}H = \{(x, y) \in X \times Y : y \in H(x)\},$$

$$\text{epi}H = \{(x, y) \in X \times Y : y \in H(x) + C\}.$$

The so-called profile mapping of H is H_+ defined by $H_+(x) = H(x) + C$. The Kuratowski-Painlevé (sequential) upper limit is defined by

$$\limsup_{x \xrightarrow{H} x_0} H(x) = \{y \in Y \mid \exists x_n \in \text{dom}H :\ x_n \to x_0, \exists y_n \in H(x_n),\ y_n \to y\},$$

where $x \xrightarrow{H} x_0$ means that $x_n \in \text{dom}H$ and $x_n \to x_0$. The Kuratowski-Painlevé lower limit is

$$\liminf_{x \xrightarrow{H} x_0} H(x) = \{y \in Y \mid \forall x_n \in \text{dom}H :\ x_n \to x_0, \exists y_n \in H(x_n),\ y_n \to y\}.$$

H is said to be compact at x_0 if any sequence $(x_n, y_n) \in \text{gr}H$ has a convergent subsequence as soon as $x_n \to x_0$.

2. Variational sets

In the sequel, if not otherwise stated, let X and Y be real normed spaces, $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $v_1, ..., v_{m-1} \in Y$.

Definition 2.1 (See [1]). The variational sets of type 1 are defined as follows:

$$V^1(F, x_0, y_0) = \limsup_{x \xrightarrow{F} x_0,\, t \to 0^+} \frac{1}{t}(F(x) - y_0), ...$$

$$V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \limsup_{x \xrightarrow{F} x_0,\, t \to 0^+} \frac{1}{t^m}(F(x) - y_0 - tv_1 - \cdots t^{m-1}v_{m-1}).$$

Definition 2.2 (See [1]). The variational sets of type 2 are defined as follows:

$$W^1(F, x_0, y_0) = \limsup_{x \xrightarrow{F} x_0} \mathrm{cone}_+(F(x) - y_0), ...,$$

$$W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \limsup_{x \xrightarrow{F} x_0\; t \to 0^+} \frac{1}{t^{m-1}}(\mathrm{cone}_+(F(x) - y_0) - v_1 - \cdots - t^{m-2}v_{m-1}).$$

By using equivalent formulations for the Kuratowski-Painlevé sequential upper limit we easily obtain the following formulae of the two types of variational sets.

Proposition 2.1 (Equivalent Formulations of V^m). $V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$ *is equal to all of the following sets*

(i) $\{y \in Y | \liminf_{x \xrightarrow{F} x_0,\, t \to 0^+} \frac{1}{t^m} d(y_0 + tv_1 + ... + t^{m-1}v_{m-1} + t^m y, F(x)) = 0\}$;

(ii) $\{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists r(t_n^m) = 0(t_n^m), \forall n, y_0 + t_n v_1 + ... + t_n^{m-1}v_{m-1} + t_n^m y + r(t_n^m) \in F(x_n)\}$;

(iii) $\{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \to y, \forall n, y_0 + t_n v_1 + ... + t_n^{m-1}v_{m-1} + t_n^m v_n \in F(x_n)\}$;

(iv) $\{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists y_n \in F(x_n), \lim_{n \to \infty} \frac{1}{t_n^m}(y_n - y_0 - t_n v_1 - ... - t_n^{m-1}v_{m-1}) = y\}$;

(v) $\bigcap_{\epsilon>0} \bigcap_{\substack{\alpha>0 \\ \beta>0}} \bigcup_{\substack{0<t\leq\alpha \\ \|x-x_0\|\leq\beta}} (\frac{1}{t^m}(F(x) - y_0 - tv_1 - ... - t^{m-1}v_{m-1}) + \epsilon B_Y);$

(vi) $\bigcap_{\substack{\alpha>0 \\ \beta>0}} \mathrm{cl} \bigcup_{\substack{0<t\leq\alpha \\ \|x-x_0\|\leq\beta}} \frac{1}{t^m}(F(x) - y_0 - tv_1 - ... - t^{m-1}v_{m-1}).$

Proposition 2.2 (Equivalent Formulations of W^m). *$W^m(F, x_0, y_0, v_1, ..., v_{m-1})$ has the following equivalent expressions*

(i) $\{y \in Y | \liminf_{x\xrightarrow{F}x_0,\, t\to 0^+} \frac{1}{t^{m-1}} d(v_1 + ... + t^{m-2}v_{m-1} + t^{m-1}y, \mathrm{cone}_+(F(x) - y_0)) = 0\};$

(ii) $\{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists r(t_n^{m-1}) = 0(t_n^{m-1}), \forall n, v_1 + ... + t^{m-2}v_{m-1} + t_n^{m-1}y + r(t_n^{m-1}) \in \mathrm{cone}_+(F(x_n) - y_0)\};$

(iii) $\{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \to y, \forall n, v_1 + ... + t^{m-2}v_{m-1} + t_n^{m-1}v_n \in \mathrm{cone}_+(F(x_n) - y_0)\};$

(iv) $\{y \in Y | \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists y_n \in \mathrm{cone}_+(F(x_n) - y_0), \lim_{n\to\infty} \frac{1}{t_n^{m-1}}(y_n - v_1 - ... - t_n^{m-2}v_{m-1}) = y\};$

(v) $\bigcap_{\epsilon>0} \bigcap_{\substack{\alpha>0 \\ \beta>0}} \bigcup_{\substack{0<t\leq\alpha \\ \|x-x_0\|\leq\beta}} [\frac{1}{t^{m-1}}(\mathrm{cone}_+(F(x) - y_0) - v_1 - ... - t^{m-2}v_{m-1}) + \epsilon B_Y];$

(vi) $\bigcap_{\substack{\alpha>0 \\ \beta>0}} \mathrm{cl} \bigcup_{\substack{0<t\leq\alpha \\ \|x-x_0\|\leq\beta}} \frac{1}{t^{m-1}}[cone_+(F(x) - y_0) - v_1 - ... - t^{m-2}v_{m-1}].$

Recall that a subset S in a linear space is called star-shaped at $x_0 \in S$ if, for all $x \in S$ and $\alpha \in [0, 1], (1 - \alpha)x_0 + \alpha x \in S$. A set-valued mapping $H : X \to 2^Y$ between two linear spaces is said to be star-shaped at $x_0 \in S$ on the star-shaped at x_0 subset $S \subseteq \mathrm{dom}H$ if, for all $x \in S$ and $\alpha \in [0, 1]$,

$$(1 - \alpha)H(x_0) + \alpha H(x) \subseteq H((1 - \alpha)x_0 + \alpha x).$$

If $C \subseteq Y$ is a cone (not necessarily convex) and we have, for all $x \in S$ and $\alpha \in [0, 1]$,

$$(1 - \alpha)H(x_0) + \alpha H(x) \subseteq H((1 - \alpha)x_0 + \alpha x) + C,$$

we say that H is C-star-shaped at x_0. When X and Y are normed, $F : X \to 2^Y$ is called pseudo-convex at $(x_0, y_0) \in \text{gr}F$ if epi$H \subseteq (x_0, y_0) + T_{\text{epi}F}(x_0, y_0)$. We have some useful properties under convexity assumptions as follows.

Proposition 2.3

(i) *If F is star-shaped at x_0, then*

$$V^1(F, x_0, y_0) = W^1(F, x_0, y_0).$$

(ii) *If we assume more that F is locally convex at (x_0, y_0) then these variational sets are convex.*

Proof. (i) Because we always have $V^m(F_i, x_0, y_0, v_1, ..., v_{m-1}) \subseteq W^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$ for all m, we need to check only the reverse containment for $m = 1$. Let v belong to the right-hand side, i.e. there are $x_n \xrightarrow{F} x_0$, $v_n \to v$, $y_n \in F(x_n)$ and $h_n > 0$ such that $v_n = h_n(y_n - y_0)$. It is clear that one can choose a sequence $t_n \to 0^+$ such that $t_n h_n \to 0^+$. Then, for n large so that $t_n h_n < 1$,

$$y_0 + t_n v_n \in F(x_0) + t_n h_n(F(x_n) - F(x_0))$$

$$\subseteq F(x_0 + t_n h_n(x_n - x_0)) := F(\overline{x_n}).$$

This means $v \in V^1(F, x_0, y_0)$.

(ii) Assume that $v_i \in W^1(F, x_0, y_0)$, i.e. there are $x_{i,n} \xrightarrow{F} x_0$, $v_{i,n} \to v_i$, $y_{i,n} \in F(x_{i,n})$ and $h_{i,n} > 0$ such that $v_{i,n} = h_{i,n}(y_{i,n} - y_0)$ for $i = 1, 2$. Then we see that

$$v_{1,n} + v_{2,n} = (h_{1,n} + h_{2,n})[(h_{1,n}y_{1,n} + h_{2,n}y_{2,n})(h_{1,n} + h_{2,n})^{-1} - y_0]$$

lies in cone$_+(F(x_n) - y_0)$ for $x_n = (h_{1,n}x_{1,n} + h_{2,n}x_{2,n})(h_{1,n} + h_{2,n})^{-1}$, for all n, by the assumed convexity. This means that the limit $v_1 + v_2$ belongs to $W^1(F, x_0, y_0)$.

□

Proposition 2.4 (See [1]). *Let* $x_0 \in S \subseteq \text{dom}F$ *and* $y_0 \in F(x_0)$. *Assume that*

(i) S *is star-shaped at* x_0 *and* F *is* C*-star-shaped at* (x_0) *on* S*; or*

(ii) F *is pseudoconvex at* (x_0, y_0).

Then, $\forall x \in S, F(x) - y_0 \subseteq V^1(F_+, x_0, y_0)$.

Therefore the following notion used later is a natural modification.

Definition 2.3. $F : X \to 2^Y$ is said to be pseudoconvex of type 1 at $(x_0, y_0) \in \text{gr}F$ if, for all $x \in \text{dom}F$, $F(x) - y_0 \subseteq V^1(F, x_0, y_0)$; and to be pseudoconvex of type 2 at (x_0, y_0) if, for all $x \in \text{dom}F$, $F(x) - y_0 \subseteq W^1(F, x_0, y_0)$.

3. Calculus of variational sets

3.1 Algebraic and set operations

As in section 2, let X and Y be real normed spaces and $v_1, ..., v_{m-1} \in Y$.

Proposition 3.1 (Union Rule). *Let* $F_i : X \to 2^Y,\ i = 1, ..., k,\ (x_0, y_0) \in \bigcup_{i=1}^{k} \text{gr}F_i$ *and* $I(x_0, y_0) = \{i \mid (x_0, y_0) \in \text{gr}F_i\}$. *Then*

(i) $V^m(\bigcup_{i=1}^{k} F_i, x_0, y_0, v_1, ..., v_{m-1}) = \bigcup_{i \in I(x_0,y_0)} V^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$;

(ii) $W^m(\bigcup_{i=1}^{k} F_i, x_0, y_0, v_1, ..., v_{m-1}) = \bigcup_{i \in I(x_0,y_0)} W^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$.

Proof. By the similarity we check only (i). Let $y \in \bigcup_{i \in I(x_0,y_0)} V^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$, $i_0 \in I(x_0, y_0)$ and $y \in V^m(F_{i_0}, x_0, y_0, v_1, ..., v_{m-1})$. There exist sequences $t_n \to 0^+$, $x_n \xrightarrow{F_{i_0}} x_0$ and $y_n \to y$ such that

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m y_n \in F_{i_0}(x_n) \subseteq \bigcup_{i=1}^{k} F_i(x_n)$$

for all n. Hence $y \in V^m(\bigcup_{i=1}^{k} F_i, x_0, y_0, v_1, ..., v_{m-1})$.

Conversely, let $y \in V^m(\bigcup_{i=1}^{k} F_i, x_0, y_0, v_1, ..., v_{m-1})$. Then there exist sequences $t_n \to 0^+$, $x_n \xrightarrow{\cup_{i=1}^{k} F_i} x_0$ and $y_n \to y$ such that

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m y_n \in \bigcup_{i=1}^{k} F_i(x_n)$$

for all n. For $i_0 \in I(x_0, y_0)$ there exist subsequence denoted the same as the supersequences, $\{x_n\} \in \mathrm{dom} F_{i_0}$ and $y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m y_n$, which lies entirely in $F_{i_0}(x_n)$. Thus

$$y \in V^m(F_{i_0}, x_0, y_0, v_1, ..., v_{m-1}) \subseteq \bigcup_{i \in I(x_0,y_0)} V^m(F_i, x_0, y_0, v_1, ..., v_{m-1}).$$

□

We omit a similar proof of the following rule.

Proposition 3.2 (Intersection Rule). *Let $F_i : X \to 2^Y$, $i = 1, ..., n$ and $(x_0, y_0) \in \bigcap_{i=1}^{n} \mathrm{gr} F_i$. Then*

(i) $V^m(\bigcap_{i=1}^{n} F_i, x_0, y_0, v_1, ..., v_{m-1}) \subseteq \bigcap_{i=1}^{n} V^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$;

(i) $W^m(\bigcap_{i=1}^{n} F_i, x_0, y_0, v_1, ..., v_{m-1}) \subseteq \bigcap_{i=1}^{n} W^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$.

Example 3.1 (Equality Fails for the Intersection Rule). Let $X = Y = \mathbb{R}$, $F_1, F_2 : X \to 2^Y$ are defined by

$$F_1(x) = \begin{cases} [-1, 1], & \text{if } x = 0, \\ \{0\}, & \text{if } x \neq 0, \end{cases}$$

$$F_2(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ [0, 1], & \text{if } x \neq 0 \end{cases}$$

and $(x_0, y_0) = (0, 0)$. Then, $V^1(F_1, 0, 0) = W^1(F_1, 0, 0) = \mathbb{R}$, $V^1(F_2, 0, 0) = W^1(F_2, 0, 0) = \mathbb{R}_+$, $W^1(F_1 \cap F_2, 0, 0) = \{0\}$, $V^1(F_1 \cap F_2, 0, 0) = W^1(F_1 \cap$

$F_2, 0, 0) = \{0\}$. However,

$$V^1(F_1, 0, 0) \cap V^1(F_2, 0, 0) = W^1(F_1, 0, 0) \cap W^1(F_2, 0, 0) = \mathbb{R}_+.$$

Example 3.2 (Equality Holds for the Intersection Rule). Let $X = Y = \mathbb{R}$, $F_1, F_2 : X \to 2^Y$ are defined by

$$F_1(x) = \begin{cases} \{0\}, & \text{if } x < 0, \\ [-1, 1], & \text{if } x = 0, \\ \{1\}, & \text{if } x > 0, \end{cases}$$

$$F_2(x) = \begin{cases} [-1, 0], & \text{if } x = 0, \\ \{1\}, & \text{if } x \neq 0, \end{cases}$$

and $(x_0, y_0) = (0, 0)$. Then, $V^1(F_1, 0, 0) = W^1(F_1, 0, 0) = W^1(F_2, 0, 0) = \mathbb{R}$ and $V^1(F_2, 0, 0) = \mathbb{R}_-$. For the intersection we have

$$(F_1 \cap F_2)(x) = \begin{cases} \emptyset, & \text{if } x < 0, \\ [-1, 0], & \text{if } x = 0, \\ \{1\}, & \text{if } x > 0, \end{cases}$$

$$V^1(F_1 \cap F_2, 0, 0) = \mathbb{R}_-, \ W^1(F_1 \cap F_2, 0, 0) = \mathbb{R}.$$

The following definition is needed for some further developments.

Definition 3.1. Let $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $v_1, ..., v_{m-1} \in Y$. If the upper limit defining $V^m(F, x_0, y_0, v_1, ..., v_{m-1})$ is a full limit, i.e. the upper limit coincides with the lower limit, then this set is called a proto-variational set of order m of type 1 of F at (x_0, y_0).

If the similar coincidence occurs for W^m we say that this set is a proto-variational set of order m of type 2 of F at (x_0, y_0).

Proposition 3.3 (Sum Rule for V^m). *Let $F_i : X \to 2^Y$, $x_0 \in \mathrm{dom}F_1 \bigcap \mathrm{int} \bigcap_{i=2}^{k} \mathrm{dom}F_i$, $y_i \in F_i(x_0)$ and $v_{i,1}, ..., v_{i,m-1} \in Y$ for $i = 1, ..., k$. If $F_i, i = 2, ...k$ have proto-variational sets $V^m(F_i, x_0, y_0, v_{i,1}, ..., v_{i,m-1})$, respectively, then*

$$\sum_{i=1}^{k} V^m(F_i, x_0, y_i, v_{i,1}, ..., v_{i,m-1}) \subseteq V^m(\sum_{i=1}^{k} F_i, x_0, \sum_{i=1}^{k} y_i, \sum_{i=1}^{k} v_{i,1}, ..., \sum_{i=1}^{k} v_{i,m-1}).$$

Proof. Consider $v_i \in V^m(F_i, x_0, y_i, v_{i,1}, ..., v_{i,m-1}), i = 1, ..., k$. One finds sequences $t_n \longrightarrow 0^+,\ x_n \xrightarrow{F_1} x_0$ and $y_{1,n} \in F_1(x_n)$ such that

$$\lim_{n\to\infty} \frac{1}{t_n^m}(y_{1,n} - y_1 - t_n v_{1,1} - ... - t_n^{m-1} v_{1,m-1}) = v_1.$$

Since $V^m(F_i, x_0, y_i, v_{i,1}, ..., v_{i,m-1}),\ i = 2, ...k$, are proto-variational sets and $x_0 \in$ int domF_i, there are $y_{i,n} \in F_i(x_n),\ i = 2, ...k$, for large n such that

$$\lim_{n\to\infty} \frac{1}{t_n^m}(y_{i,n} - y_i - t_n v_{i,1} - ... - t_n^{m-1} v_{i,m-1}) = v_i.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{t_n^m}(\sum_{i=1}^k y_{i,n} - \sum_{i=1}^k y_i - t_n \sum_{i=1}^k v_{i,1} - ... - t_n^{m-1} \sum_{i=1}^k v_{i,m-1}) = \sum_{t=1}^k v_i.$$

Since the left-hand side of the last equality belongs to the right-hand side of the required inclusion, we are done. □

We cannot reduce the condition $x_0 \in \text{dom}F_1 \bigcap \text{int} \bigcap_{i=2}^k \text{dom}F_i$ to $x_0 \in \bigcap_{i=1}^k \text{dom}F_i$ as illustrated by

Example 3.3. Let $X = Y = \mathbb{R}$, $x_0 = y_1 = y_2 = 0$ and $F_1, F_2 : X \to 2^Y$ be defined by

$$F_1(x) = \begin{cases} \mathbb{R}_+, & \text{if } x \geq 0, \\ \emptyset, & \text{if } x < 0, \end{cases}$$

$$F_2(x) = \begin{cases} \mathbb{R}_-, & \text{if } x < 0, \\ \{0\}, & \text{if } x = 0, \\ \emptyset, & \text{if } x > 0, \end{cases}$$

Then, $V^1(F_1, 0, 0)$ is a proto-variational set and

$$V^1(F_1, 0, 0) + V^1(F_2, 0, 0) = \mathbb{R},$$

$$V^1(F_1 + F_2, 0, 0) = \mathbb{R}_+.$$

Furthermore, the following example explains, unfortunately, that W^m does not satisfy the rule similar to Proposition 3.3 even for $m = 1$. However, here a reverse containment is true for W^1. It is interesting that this reverse containment holds for W^1 in a general case as shown in Proposition 3.4 below.

Example 3.4 Let X = Y = $\mathbb{R}$, $x_0 = 0$, $y_1 = 1$, $y_2 = -1$ and $F_1, F_2 : X \to 2^Y$ be defined by

$$F_1(x) = \begin{cases} \{1\} & \text{if } \ x = 0, \\ \{0,1\} & \text{if } \ x \neq 0, \end{cases}$$

$$F_2(x) = \begin{cases} [-1,+\infty) & \text{if } \ x = 0, \\ \mathbb{R}_+ & \text{if } \ x \neq 0. \end{cases}$$

Then

$$(F_1 + F_2)(x) = \mathbb{R}_+, \forall x \in \mathbb{R},$$

$$W^1(F_1, x_0, y_1) = \mathbb{R}_-,$$

$$W^1(F_2, x_0, y_2) = \mathbb{R}_+,$$

$$W^1(F_1 + F_2, x_0, y_1 + y_2) = \mathbb{R}_+.$$

and we have a containment strict reverse to that asserted in Proposition 3.3, although F_2 has proto-variational set of order 1 of type 2 at (x_0, y_2). We also see that this containment holds (not by chance, since the compactness required in Proposition 3.4 below is satisfied).

Proposition 3.4 (Sum Rule for W^1). *Let $F_i : X \to 2^Y$, $(x_0, y_i) \in \mathrm{gr}F_i$ and F_i be compact at x_0 for $i = 1, ..., k$. Then*

$$\sum_{i=1}^{k} W^1(F_i, x_0, y_i) \supseteq W^1(\sum_{i=1}^{k} F_i, x_0, \sum_{i=1}^{k} y_i).$$

Proof. For the sake of simplicity we discuss only the case $k = 2$ (the same is for general k). Let $y \in W^1(F_1 + F_2, x_0, y_1 + y_2)$, $x_n \xrightarrow{F_1+F_2} x_0$, $y_n \to y$, $h_n > 0$ and

$$y_n \in \frac{1}{h_n} \sum_{i=1}^{2} (F_i(x_n) - y_i)$$

for all n. Then there exists $\overline{y_{i,n}} \in F_i(x_n)$ such that

$$h_n y_n = \sum_{i=1}^{2} (\overline{y_{i,n}} - y_i).$$

Since F_1 and F_2 are compact, there exist two subsequences (the subscripts of the second one are taken among those of the first), denoted by the same notation $\overline{y_{i,n}}$, which converge to $\overline{y_i}$, respectively, for $i = 1, 2$. Consequently, h_n also tends to some nonnegative number h and we have in the limit

$$y = \frac{1}{h}[(\overline{y_1} - y_1) + (\overline{y_2} - y_2)].$$

Observing that $\overline{y_{i,n}} - y_i \in F_i(x_n) - y_i$ for all n, which means $\overline{y_i} - y_i \in W^1(F_i, x_0, y_i)$, and $W^1(F_i, x_0, y_i)$ is a cone, the last equality completes the proof. $\square$

Unfortunately, the similar rule is not true for V^1 as indicated by the example below, which says also that the proto-variationality assumed in Proposition 3.3 cannot be dropped.

Example 3.5. Let $X = Y = \mathbb{R}$, $x_0 = 0, y_1 = 0, y_2 = 1$ and $F_1, F_2 : X \to 2^Y$ be defined by

$$F_1(x) = \begin{cases} [0,1], & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0, \end{cases}$$
$$F_2(x) = \begin{cases} \{0\}, & \text{if } x \neq 0, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Then,

$$V^1(F_1, 0, 0) = R_+, \quad V^1(F_2, 0, 1) = \{0\},$$
$$(F_1 + F_2)(x) = \begin{cases} [0,1], & \text{if } x \neq 0, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

We see that $V^1(F_1 + F_2, 0, 0 + 1) = \mathbb{R}_-$ is incomparable with the sum of the two variational sets, although both F_1 and F_2 are compact at x_0 as required for W^1 in Proposition 3.4. The inclusion of Proposition 3.3 does not hold as neither F_1 nor F_2 has a proto variational set of type 1 at (x_0, y_0).

The following result can be validated similarly as Proposition 3.3.

Proposition 3.5 (Descartes Product). *Let $F_i : X_i \to 2^{Y_i}$, $x_i \in \mathrm{dom} F_i$, $y_i \in F_i(x_i)$ and $v_{i,1}, ..., v_{i,m-1} \in Y_i$ for $i = 1, ..., k$. Then*

(i) $$\prod_{i=1}^{k} V^m(F_i, x_i, y_i, v_{i,1}, ..., v_{i,m-1})$$
$$\supseteq V^m(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k), (v_{1,1}, ..., v_{1,m-1}), ..., (v_{k,1}, ..., v_{k,m-1})),$$
$$\prod_{i=1}^{k} W^m(F_i, x_i, y_i, v_{i,1}, ..., v_{i,m-1})$$
$$\supseteq W^m(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k), (v_{1,1}, ..., v_{1,m-1}), ..., (v_{k,1}, ..., v_{k,m-1}));$$

(ii) *if $F_2, .., F_k$ have proto-variational sets of type 1 and $x_0 \in \mathrm{dom} F_1 \bigcap \mathrm{int} \bigcap_{i=2}^{k} \mathrm{dom} F_i$, then the containment for V^m in (i) becomes equality.*

The following example says that even for $m = 1$ the counterpart of Proposition 3.5 (ii) for W^1 is not true.

Example 3.6 Let $X = Y = \mathbb{R}$, $F_1, F_2 : X \to 2^Y$ are defined by

$$F_1(x) = \begin{cases} \{1\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0, \end{cases}$$

$$F_2(x) = \begin{cases} \{0\}, & \text{if } x \neq 0, \\ \{0, 1\}, & \text{if } x = 0. \end{cases}$$

Then, F_2 has a proto-variational set of order 1 of type 2 at 0 and one has by direct computations

$$(F_1 \times F_2)(x_1, x_2) = \begin{cases} \{(1, 0)\}, & \text{if } x_1 \neq 0,\ x_2 \neq 0, \\ \{(1, 0), (1, 1)\}, & \text{if } x_1 \neq 0,\ x_2 = 0, \\ \{(0, 0)\}, & \text{if } x_1 = 0,\ x_2 \neq 0, \\ \{(0, 0), (0, 1)\}, & \text{if } (x_1, x_2) = (0, 0), \end{cases}$$

$$W^1(F_1, 0, 0) = \mathbb{R}_+,\ W^1(F_2, 0, 1) = \mathbb{R}_-,$$

$$W^1(F_1 \times F_2, (0, 0), (0, 1)) = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_-) \cup \{(y, -y) :\ y \geq 0\}.$$

Hence, $W^1(F_1 \times F_2, (0,0), (0,1))$ is strictly included in $W^1(F_1, 0, 0) \times W^1(F_2, 0, 1)$.

Moreover, assertion (ii) is not a necessary condition even with $m = 1$ for the equality to hold for both V^1 and W^1 as shown by the next result.

Proposition 3.6 (Descartes Product for V^1). *Let $F_i : X_i \to 2^{Y_i}$ be star-shaped at x_i, $x_i \in \mathrm{dom} F_i$ and $y_i \in F_i(x_i)$ for $i = 1, ..., k$. Then*

$$\prod_{i=1}^{k} V^1(F_i, x_i, y_i) = V^1(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k)),$$

$$\prod_{i=1}^{k} W^1(F_i, x_i, y_i) = W^1(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k)).$$

Proof. First, for V^1 we have to check only the inclusion $\subseteq$. Let $(z_1, ..., z_k) \in \prod_{i=1}^{k} V^1(F_i, x_i, y_i)$. Then one has sequences $1 > t_{i,n} \to 0^+$, $x_{i,n} \xrightarrow{F_i} x_i$ and $y_{i,n} \in F_i(x_{i,n})$ for $i = 1, ..., k$ such that

$$\lim_{n\to\infty} z_{i,n} := \lim_{n\to\infty} \frac{1}{t_n^m}(y_{i,n} - y_i) = z_i.$$

Setting $t_n = (\prod_{i=1}^{k} t_{i,n})(\sum_{i=1}^{k} t_{i,n})^{-1}$ one sees that, for $i = 1, ..., k$,

$$\begin{array}{rcl} y_i + t_n z_{i,n} & = & y_i + \frac{t_n}{t_{i,n}}(y_{i,n} - y_i) \\ & \in & F_i(x_i) + \frac{t_n}{t_{i,n}}(F_i(x_{i,n}) - F_i(x_i)) \\ & \subseteq & F_i(x_i + \frac{t_n}{t_{i,n}}(x_{i,n} - x_i)) \end{array}$$

(the last inclusion is due to the star-shape of F_i). Now one obtains sequences $t_n \to 0^+$, $\overline{x_{i,n}} := x_i + \dfrac{t_n}{t_{i,n}}(x_{i,n} - x_i) \xrightarrow{F_i} x_i$ and $\overline{y_{i,n}} := y_i + t_n z_{i,n} \in F(\overline{x_{i,n}})$ for $i = 1, ..., k$. This means $(z_1, ..., z_k) \in V^1(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k))$.

Now for W^1, by the definition of V^1, W^1; Proposition 3.5 (i) and Proposition 2.3 (i) one has

$$V^1(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k)) \subseteq W^1(\prod_{i=1}^{k} F_i, (x_1, ..., x_k), (y_1, ..., y_k))$$

$$\subseteq \prod_{i=1}^{k} W^1(F_i, x_i, y_i) \subseteq \prod_{i=1}^{k} V^1(F_i, x_i, y_i).$$

□

The following example explains that the star-shape cannot be dispensed within the preceding statement.

Example 3.7 Let $X = Y = \mathbb{R}$, $F_1, F_2 : X \to 2^Y$ are defined by

$$F_1(x) = \begin{cases} \{1\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0, \end{cases}$$

$$F_2(x) = \begin{cases} \{-1\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

Then,

$$(F_1 \times F_2)(x) = \begin{cases} \{(1,-1)\}, & \text{if } x_1 \neq 0,\ x_2 \neq 0, \\ \{(1,0)\}, & \text{if } x_1 \neq 0,\ x_2 = 0, \\ \{(0,-1)\}, & \text{if } x_1 = 0,\ x_2 \neq 0, \\ \{(0,0)\}, & \text{if } (x_1, x_2) = (0,0), \end{cases}$$

$$W^1(F_1, 0, 0) = \mathbb{R}_+,\ W^1(F_2, 0, 0) = \mathbb{R}_-,$$

$$W^1(F_1 \times F_2, (0,0), (0,0)) = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_-) \cup \{(y, -y) :\ y \geq 0\}.$$

Hence, $W^1(F_1, 0, 0) \times W^1(F_2, 0, 0)$ is not included in $W^1(F_1 \times F_2, (0,0), (0,0))$. The reason is the lack of the required star-shape.

3.2 Compositions

For $F : X \to 2^Y$ and $G : Y \to 2^Z$ we have two compositions as follows

$$(G \circ F)(x) = \bigcup \{G(y) |\ y \in F(x)\},$$

$$(G \Box F)(x) = \bigcap \{G(y) |\ y \in F(x)\}.$$

Proposition 3.7 (Chain Rule for V^m). *Let* $F : X \to 2^Y,\ G : Y \to 2^Z,\ (x_0, y_0) \in \mathrm{gr}F,\ (y_0, z_0) \in \mathrm{gr}G$ *and* $\mathrm{im}F \subseteq \mathrm{dom}G$.

(i) *If* G *is Lipschitz around* y_0 *then, for* $u_1 \in V^1(F, x_0, y_0), ..., u_{m-1} \in V^{m-1}(F, x_0, y_0, u_1, ..., u_{m-2})$ *and* $v_1 \in D^bG(y_0, z_0)(u_1), ..., v_{m-1} \in D^{b(m-1)}G(y_0, z_0, v_1, ..., v_{m-2})(u_{m-1})$, *we have*

$$D^{b(m)}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(V^m(F, x_0, y_0, u_1, ..., u_{m-1}))$$

$$\subseteq V^m(G \circ F, x_0, z_0, v_1, ..., v_{m-1}).$$

(ii) *If additionally* F *has a proto-variational set of order* m *of type 1 at* (x_0, y_0), *then*

$$D^mG(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(V^m(F, x_0, y_0, u_1, ..., u_{m-1}))$$

$$\subseteq V^m(G \circ F, x_0, z_0, v_1, ..., v_{m-1}).$$

(iii) *If* F *is l.s.c. at* (x_0, y_0) *then* $V^m(G\Box F, x_0, z_0, v_1, ..., v_{m-1}) \subseteq V^m(G, y_0, z_0, v_1, ..., v_{m-1})$.

Proof. (i) Let $z \in D^{b(m)}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(V^m(F, x_0, y_0, u_1, ..., u_{m-1}))$. There exists $v \in V^m(F, x_0, y_0, u_1, ..., u_{m-1})$ such that $z \in D^{b(m)}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(v)$. Hence, for v, there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$ and $v_n \to v$ such that

$$y_0 + t_nu_1 + ... + t_n^{m-1}u_{m-1} + t_n^m v_n \in F(x_n).$$

With t_n above, for z there exists $(\overline{v_n}, \overline{z_n}) \to (v, z)$ such that

$$z_0 + t_nv_n + ... + t_n^{m-1}v_{m-1} + t_n^m\overline{z_n} \in G(y_0 + t_nu_1 + ... + t_n^{m-1}u_{m-1} + t_n^m\overline{v_n}).$$

Since G is Lipschitz around y_0, for large n one has $l > 0$ such that

$$G(y_0 + t_nu_1 + ... + t_n^{m-1}u_{m-1} + t_n^m\overline{v_n})$$

$$\subseteq G(y_0 + t_nu_1 + ... + t_n^{m-1}u_{m-1} + t_n^m v_n) + l\|t_n^m(\overline{v_n} - v_n)\|B_Z.$$

Consequently, there exists $b \in B_Z$ such that

$$z_0 + t_nv_1 + ... + t_n^{m-1}v_{m-1} + t_n^m[\overline{z_n} - l\|\overline{v_n} - v_n\|b] \in (G \circ F)(x_n)$$

and $\overline{z_n} - l\|\overline{v_n} - v_n\|b \to z$. Thus $z \in V^m(G_oF, x_0, z_0, v_1, ..., v_{m-1})$.

(ii) Let $z \in D^{(m)}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(V^m(F, x_0, y_0, u_1, ..., u_{m-1}))$. Then there exists $v \in V^m(F, x_0, y_0, u_1, ..., u_{m-1})$ such that $z \in D^{(m)}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(v)$. Since $V^m(F, x_0, y_0, u_1, ..., u_{m-1})$ is a proto-variational set of F of order m of type 1 at (x_0, y_0), for all sequences $t_n \to 0^+$ and $x_n \xrightarrow{F} x_0$, there exists a sequence $v_n \to v$ such that

$$y_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m v_n \in F(x_n).$$

as $z \in D^{(m)}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(v)$, there exists $\overline{t_n} \to 0^+$ and $(\overline{v_n}, \overline{z_n}) \to (v, z)$ satisfying

$$z_0 + t_n v_n + ... + t_n^{m-1} v_{m-1} + t_n^m \overline{z_n} \in G(y_0 + t_n u_1 + ... + t_n^{m-1} + t_n^m \overline{v_n}).$$

The rest of the proof is the same as for (i).

(iii) Let $w \in V^m(G\square F, x_0, z_0, v_1, ..., v_{m-1})$. Then there exist sequences $t_n \to 0^+$, $x_n \xrightarrow{G\square F} x_0$ and $w_n \to w$ such that, for all n,

$$z_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m w_n \in (G\square F)(x_n),$$

that is, for all $y_n \in F(x_n)$,

$$z_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m w_n \in G(y_n).$$

Since F is lsc and $y_0 \in F(x_0), x_n \to x_0$, there exists $\overline{y_n} \in F(x_n)$ such that $\overline{y_n} \to y_0$. Hence

$$z_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m w_n \in G(\overline{y_n}),$$

i.e. $w \in V^m(G, y_0, z_0, v_1, ..., v_{m-1})$. □

Proposition 3.8 (Chain Rule for W^m). *Let* $F : X \to 2^Y,\ G : Y \to 2^Z,\ (x_0, y_0) \in \mathrm{gr}F,\ (y_0, z_0) \in \mathrm{gr}G$ *and* $\mathrm{im}F \subseteq \mathrm{dom}G$.

(i) *If F is star-shaped at x_0 and G is Lipschitz around y_0, then*

$$D^bG(y_0,z_0)[W^1(F,x_0,y_0)] \subseteq DG(y_0,z_0)[W^1(F,x_0,y_0)] \subseteq V^1(G\circ F,x_0.z_0).$$

(ii) *If F is l.s.c. at (x_0,y_0) then*

$$W^m(G\Box F,x_0,z_0,v_1,...,v_{m-1}) \subseteq W^m(G,y_0,z_0,v_1,...,v_{m-1}).$$

Proof. (i) The first inclusion is obvious. For the second one let $v \in DG(y_0,z_0)(u)$ with $u \in W^1(F,x_0,y_0)$. Then, for u there exist sequences $x_n \xrightarrow{F} x_0$, $u_n \to u$, $h_n > 0$ and $y_n \in F(x_n)$ such that $u_n = h_n(y_n - y_0)$ for each n. For v one has sequences $t_n \to 0^+$ and $(a_n,b_n) \to (u,v)$ such that $z_0 + t_nb_n \in G(y_0 + t_na_n)$ for all n. We extract a subsequence of t_n by putting $\overline{t_s} = t_{n_s}$, where

$$n_1 = \min\{n \in \mathbb{N} \mid \ t_nh_1 < 1\},...,$$

$$n_s = \min\{n \in n_{s-1} + \mathbb{N} \mid \ t_nh_s < 1\}.$$

We also use the corresponding subsequences $\overline{a_s} = a_{n_s}$ and $\overline{b_s} = b_{n_s}$. In virtue of the assumed star-shapedness one has

$$y_0 + \overline{t_n}u_n = y_0 + \overline{t_n}h_n(y_n - y_0)$$

$$\in F(x_0) + \overline{t_n}h_n(F(x_n) - F(x_0)) \subseteq F(x_0 + \overline{t_n}h_n(x_n - x_0)) := F(\overline{x_n}).$$

By the Lipschitz continuity of G, there exists $L > 0$ such that, for n large enough,

$$z_0 + \overline{t_n}\overline{b_n} \in G(y_0 + \overline{t_n}\overline{a_n}) \subseteq G(y_0 + \overline{t_n}u_n) + L\overline{t_n}\|\overline{a_n} - u_n\|B_Z$$

$$\subseteq (G\circ F)(\overline{x_n}) + Lt_n\|\overline{a_n} - u_n\|B_Z.$$

Hence, for some $b \in B_Z$ and all n,

$$z_0 + \overline{t_n}(\overline{b_n} - L\|\overline{a_n} - u_n\|b) \in (G\circ F)(\overline{x_n}).$$

Therefore, $v \in V^1(G\circ F,x_0,z_0)$, as $\overline{b_n} - L\|\overline{a_n} - u_n\|b \to v$.

(ii) It is analogous to the proof of (iii) of Proposition 3.7. □

For a special case where $G = g$ is single-valued we have the following chain rule of first and second orders, which provides relations between direct images of variational sets of first and second orders and the corresponding variational sets of the images of the mappings in question.

Proposition 3.9 (Composition with Differentiable Map). *Let* $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$, $g : Y \to Z$ *be differentiable at* y_0.

(i) $\mathrm{cl}\,[\bigcup_{y\in g^{-1}(z)\cap F(x)} g'(y)V^1(F, x, y)] \subseteq V^1(g \circ F, x, z)$.

(ii) *If* $g''(y_0)$ *exists then, for all* $v_1 \in Y$,

$$g'(y_0)[V^2(F, x_0, y_0, v_1)] \subseteq V^2(g \circ F, x_0, g(y_0), g'(y_0)v_1) - \frac{1}{2}g''(y_0)(v_1, v_1).$$

Proof. (i) Let $v \in V^1(F, x_0, y_0)$ and sequences $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$ and $v_n \to v$ satisfy $y_0 + t_nv_n \in F(x_n)$ for all n. Then $g(y_0 + t_nv_n) \in (g \circ F)(x_n)$. On the other hand,

$$g(y_0 + t_nv_n) = g(y_0) + t_n(g'(y_0)v_n + \frac{0(t_n)}{t_n}).$$

Hence $g'(y_0)v \in V^1(g \circ F, x_0, g(y_0))$. Since the latter object is a closed cone, we arrive at the required inclusion.

(ii) Let $v_2 \in V^2(F, x_0, y_0, v_1)$, $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$ and $v_{2n} \to v_2$ be such that, for all n, $y_0 + t_nv_1 + t_n^2v_{2n} \in F(x_n)$ and hence

$$g(y_0 + t_nv_1 + t_n^2v_{2n}) \in (g \circ F)(x_n).$$

By the Taylor expansion,

$$g(y_0+t_nv_1+t_n^2v_{2n}) = g(y_0)+t_ng'(y_0)v_1+t_n^2\left[\frac{1}{2}g''(y_0)(v_1, v_1) + g'(y_0)v_{2n} + \vartheta(t_n)\right].$$

Therefore,

$$\frac{1}{2}g''(y_0)(v_1, v_1) + g'(y_0)v_2 \in V^2(g \circ F, x_0, z_0, g'(y_0)v_1).$$

□

The inclusion in Proposition 3.9 (i) becomes equality under lower semicontinuity and calmness assumptions as follows.

Proposition 3.10 (Equality in Composition with Differentiable Map). *Let Y be finite dimensional, $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $g : Y \to Z$. Assume that*

(i) *F is l.s.c. at (x_0, y_0);*

(ii) *g is differentiable at y_0;*

(iii) *the map $g^{-1} : (g \circ F)(x_0) \to 2^{F(x_0)}$ defined by $z \mapsto g^{-1}(z) \cap F(x_0)$ satisfies the calmness property: for some $l > 0$ and all z in a neighborhood of $g(y_0)$,*

$$d(y_0, g^{-1}(z) \cap F(x_0)) \le \|z - g(y_0)\|.$$

Then

$$\mathrm{cl}(g'(y_0)V^1(F, x_0, y_0)) = V^1(g \circ F, x_0, g(y_0)).$$

Proof. We need to prove only $g'(y_0)V^1(F, x_0, y_0) \supseteq V^1(g \circ F, x_0, g(y_0))$. For $y \in V^1(g \circ F, x_0, g(y_0))$, there exist sequences $t_n \to 0^+$, $x_n \overset{g \circ F}{\to} x_0$, $v_n \to y$ such that $g(y_0) + t_n v_n \in g \circ F(x_n)$ for all n. By the calmness assumption, for large n,

$$d(y_0, g^{-1}(z_n) \cap F(x_0)) \le l\|z_n - f(y_0)\|.$$

Hence, for $\epsilon > 0$, there is $y_n \in g^{-1}(z_n) \cap F(x_0)$ such that, for $u_n := \frac{1}{t_n}(y_n - y_0)$,

$$\|u_n\| \le (l + \epsilon)\|v_n\|.$$

Therefore, we have a subsequence, denoted also by u_n, which converges to some u. This results in $u \in V^1(F, x_0, y_0)$, since by the lower semicontinuity of F one has, for large n,

$$y_0 + t_n u_n = y_n \in g^{-1}(g \circ F)(x_n)) \bigcap F(x_0) \subseteq F(x_n).$$

Observing that $v_n = \frac{1}{t_n}(g(y_0 + t_n u_n) - g(y_0))$ tends to $g'(y_0)u$ we conclude $y \in f'(y_0)V^1(F, x_0, y_0)$, since we know that $v_n \to v$. □

For a more specific case of Propositions 3.9 and 3.10 where $g \in L(Y,Z)$, we have similar results for all $m \in \mathbb{N}$, not only for $m=1$, as follows.

Proposition 3.11 (Composition with Linear Continuous Map). *Let $F : X \to 2^Y, x \in \text{dom}F$ and $g \in L(Y,Z)$. Then*

(i) *for any $m \in \mathbb{N}$ there holds*

$$\text{cl}[\bigcup_{y \in g^{-1}(z) \cap F(x)} g(V^m(F,x,y,v_1,...,v_{m-1}))] \subseteq V^m(g \circ F, x, z, g(v_1),...,g(v_{m-1})).$$

If additionally F is pseudoconvex of type 1 at $(x_0,y_0) \in$grF, then one has equality for $m=1$;

(ii) *for all $m \in \mathbb{N}$ one has*

$$\text{cl}[\bigcup_{y \in g^{-1}(z) \cap F(x)} g(W^m(F,x,y,v_1,...,v_{m-1}))] \subseteq W^m(g \circ F, x, z, g(v_1),...,g(v_{m-1})).$$

If additionally F is pseudoconvex of type 1 at $(x_0,y_0) \in \text{gr}F$, then one has equality for $m=1$.

Proof. (i) For each $y \in g^{-1}(z) \cap F(x)$ we have

$$g[V^m(F,x,y,v_1,...,v_{m-1})] = g\left[\limsup_{x' \xrightarrow{F} x,\, t \to 0^+} \frac{1}{t^m}(F(x) - y - tv_1 - ... - t^{m-1}v_{m-1})\right]$$

$$\subseteq \limsup_{x' \xrightarrow{F} x,\, t \to 0^+} \frac{1}{t^m}((g \circ F)(x) - z - tg(v_1) - ... - t^{m-1}g(v_{m-1}))$$

$$= V^m(g \circ F, x, g(y_0), g(v_1),...,g(v_{m-1}))$$

(the inclusion is due to Theorem 4.26 of [7] and the linearity of g). By the closedness of the variational set we are done.

If F is pseudoconvex of type 1 at (x_0,y_0) and $x_n \in \text{dom}F$, for $y \in V^1(g \circ F, x_0, g(y_0))$, there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$ and $y_n \to y$ such that

$$g(y_0) + t_n y_n \in (g \circ F)(x_n) \subseteq g(V^1(F,x_0,y_0)) + g(y_0).$$

Hence,

$$y_n \in \frac{1}{t_n} g(V^1(F, x_0, y_0) \subseteq g(V^1(F, x_0, y_0))$$

and thus $y \in \overline{g[V^1(F, x_0, y_0)]}$.

(ii) The assertion for W^m can be checked by Theorem 4.26 of [7] as for V^m but we give a simple direct proof. Let $y \in W^m(F, x_0, y_0, v_1, ..., v_{m-1})$ and $x \xrightarrow{F} x_0$, $t_n \to 0^+$ and $y_n \to y$ with

$$v_1 + ... + t_n^{m-2} v_{m-1} + t_n^{m-1} y_n \in \text{cone}_+(F(x_n) - y_0).$$

for all n. By the linearity of g one has

$$g(v_1) + ... + t_n^{m-2} g(v_{m-1}) + t_n^{m-1} g(y_n) \in \text{cone}_+[(g \circ F)(x_n) - g(y_0)].$$

Therefore, $g(y) \in W^m(g \circ F, x_0, g(y_0), g(v_1), ..., g(v_{m-1}))$. The pseudoconvex case is proved similarly as in (i). □

In the case where Y is finite dimensional, for $m = 1$ we can obtain the equality in the conclusion of the preceding proposition under a condition on $\ker(g)$ (the null space of g) instead of the pseudoconvexity assumption. We need the following definition of the horizon upper limit of $F : X \to Y$ in [7]

$$\text{limsup}^{\infty}_{x \xrightarrow{F} x_0} F(x) = \{y \in Y \,|\, \exists x_n \xrightarrow{F} x_0, \exists \lambda_n \to 0^+, \exists y_n \in F(x_n), \lambda_n y_n \to y\}.$$

Proposition 3.12. *Let $F : X \to 2^Y$, $g \in L(Y, Z)$, $x \in \text{dom}F$ and $z \in Z$. Let Y be finite dimensional and $y \in g^{-1}(z) \cap F(x)$.*

(i) *If*

$$\ker(g) \bigcap \text{limsup}^{\infty}_{x' \xrightarrow{F} x,\, t \to 0^+} \frac{1}{t}(F(x') - y) = \{0\},$$

then

$$\text{cl}[\bigcup_{y \in g^{-1}(z) \cap F(x)} g(V^1(F, x, y))] \subseteq V^1(g \circ F, x, z).$$

(ii) *If*

$$\ker(g)\bigcap W^1(F,x,y)=\{0\},$$

then

$$\mathrm{cl}[\bigcup_{y\in g^{-1}(z)\cap F(x)} g(W^1(F,x,y))]\subseteq W^1(g\circ F,x,z).$$

Proof. We demonstrate only (i), since (ii) is similar and simpler. Only the containment $\supseteq$ needs to be considered. Let u belong to the right-hand side, i.e. for some sequences $t_n\to 0^+$, $x_n\xrightarrow{F} x$, $u_n\to u$ and $y_n\in g(y_n)$ one has $z+t_nu_n\in g(y_n)$ for all n. Set $v_n=\frac{1}{t_n}(y_n-y)$. If $\{v_n\}$ is bounded then one can assume that v_n tends to some v, which satisfies $v\in V^1(F,x,y)$ and $g(v)=u$ as required. So it remains to check this boundedness. Suppose $\|v_n\|\to\infty$ and set $\overline{v_n}=\frac{v_n}{\|v_n\|}$ which is assumed to have a limit $\overline{v}$ with norm one. Then $g(\overline{v})=0$. Furthermore $\overline{v}\in\mathrm{limsup}^{\infty}_{x'\xrightarrow{F}x,\ t\to 0^+}\frac{1}{t}(F(x')-y)$, which is impossible. □

For the following special case equality holds for $m=1$ without any assumption.

Corollary 3.13. *Let* $F:X\to 2^Y,\ (x_0,y_0)\in\mathrm{gr}F$ *and* $\lambda\in\mathbb{R}$.

(i) $\lambda V^m(F,x_0,y_0,v_1,...,v_{m-1})\subseteq V^m(\lambda F,x_0,\lambda y_0,\lambda v_1,..,\lambda v_{m-1})$. *The equality always holds for* $m=1$.

(i) $\lambda W^m(x_0,y_0,v_1,...,v_{m-1})\subseteq W^m(\lambda F,x_0,\lambda y_0,\lambda v_1,..,\lambda v_{m-1})$. *The equality always holds for* $m=1$.

For scaling only the directions $v_1,...,v_{m-1}$ we easily demonstrate by definition the following rule.

Proposition 3.14 (Scaling the Directions). *Let* $F:X\to 2^Y,\ (x_0,y_0)\in\mathrm{gr}F,\ \lambda>0$ *and* $v_1,...,v_{m-1}\in Y$. *Then*

(i) $V^m(F, x_0, y_0, \lambda v_1, ..., \lambda^{m-1} v_{m-1}) = \lambda^m V^m(F, x_0, y_0, v_1, ..., v_{m-1})$;

(ii) $W^m(F, x_0, y_0, v_1, ..., \lambda^{m-2} v_{m-1}) = \lambda^{m-1} W^m(F, x_0, y_0, v_1, ..., v_{m-1})$.

3.3 More calculus

Now we analyze calculus rules for the following operations.

Definition 3.7 (See [8])

(i) For $F_1, F_2 : X \to 2^{\mathbb{R}^m}$ the inner product $\langle F_1, F_2 \rangle$ of F_1 and F_2 is the multi-function $\langle F_1, F_2 \rangle : X \to 2^{\mathbb{R}}$ defined by

$$\langle F_1, F_2 \rangle(x) = \bigcup_{y_1 \in F_1(x), y_2 \in F_2(x)} \langle y_1, y_2 \rangle.$$

(ii) For $F_1, F_2 : X \to 2^{\mathbb{R}^m}$ the outer product $F_1 \diamond F_2$ of F_1 and F_2 is the multi-function $F_1 \diamond F_2 : X \to 2^{M_m}$ defined by

$$(F_1 \diamond F_2)(x) = \bigcup_{y_1 \in F_1(x), y_2 \in F_2(x)} y_1 \diamond y_2,$$

where M_m is the space of the $m \times m$-matrices and $y_1 \diamond y_2$ is the outer product defined after Definition 3.7.

(iii) For $F_1, F_2 : X \to 2^{\mathbb{R}}$ the fraction F_1/F_2 has the values

$$(F_1/F_2)(x) = \bigcup_{y_1 \in F_1(x), y_2 \in F_2(x)} \{y_1/y_2,\ y_2 \neq 0\}.$$

(iv) For $F_1, F_2 : X \to 2^{\mathbb{R}}$ the maximum $F_1 \vee F_2$ of F_1 and F_2 is a multifunction defined by

$$(F_1 \vee F_2)(x) = \{z \in \mathbb{R} \mid \exists y_1 \in F_1(x), \exists\, y_2 \in F_2(x)\ :\ \max\{y_1, y_2\} = z\}.$$

(v) For $F_1, F_2 : X \to 2^{\mathbb{R}}$ the minimum $F_1 \wedge F_2$ has the values

$$(F_1 \wedge F_2)(x) = \{z \in \mathbb{R} \mid \exists y_1 \in F_1(x), \exists\, y_2 \in F_2(x)\ :\ \min\{y_1, y_2\} = z\}.$$

We recall that, for $u, v \in \mathbb{R}^m$, the outer product is the $m \times m$-matrix

$$u \diamond v = \begin{pmatrix} u_1v_1 & u_1v_2 & \dots & u_1v_m \\ u_2v_1 & u_2v_2 & \dots & u_2v_m \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \dots & u_mv_m \end{pmatrix}.$$

Proposition 3.15 (Inner Product Rule). *Let $x_0 \in \mathrm{dom}F_1 \cap \mathrm{intdom}F_2$ and $z_0 \in \langle F_1, F_2\rangle(x_0)$ with $z_0 = \langle y_1, y_2\rangle$ for $y_1 \in F_1(x_0)$ and $y_2 \in F_2(x_0)$.*

(i) *If F_2 has proto-variational set $V^1(F_2, x_0, y_2)$, then*

$$\langle y_1, V^1(F_2, x_0, y_2)\rangle + \langle y_2, V^1(F_1, x_0, y_1)\rangle \subseteq V^1(\langle F_1, F_2\rangle, x_0, z_0).$$

(ii) *If F_2 has proto-variational set $V^2(F_2, x_0, y_2, v_2^1)$, then*

$$\langle y_1, V^2(F_2, x_0, y_2, v_2^1)\rangle + \langle y_2, V^2(F_1, x_0, y_1, v_1^1)\rangle$$

$$\subseteq V^2(\langle F_1, F_2\rangle, x_0, z_0, \langle y_1, v_2^1\rangle + \langle y_2, v_1^1\rangle) - \langle v_1^1, v_2^1\rangle.$$

Proof. If $v_1^1 = v_2^1 = 0$, (ii) collapses to (i). To demonstrate (ii) assume that $v_1^2 \in V^2(F_1, x_0, y_1, v_1^1)$ and $v_2^2 \in V^2(F_2, x_0, y_2, v_2^1)$. For v_1^2 there exist sequences $t_n \to 0^+, x_n \xrightarrow{F_1} x_0$ and $v_{1,n} \to v_1^2$ such that $y_1 + t_nv_1^1 + t_n^2v_{1,n} \in F_1(x_n)$ for all n. For v_2^2 and the above sequences t_n and x_n, there exists $v_{2,n} \to v_2^2$ such that $y_2 + t_nv_2^1 + t_n^2v_{2,n} \in F_2(x_n)$. Therefore, for all n, the following number is in $\langle F_1, F_2\rangle(x_n)$

$\langle y_1 + t_nv_1^1 + t_n^2v_{1,n}, y_2 + t_nv_2^1 + t_n^2v_{2,n}\rangle = \langle y_1, y_2\rangle + t_n[\langle y_1, v_2^1\rangle + \langle y_2, v_1^1\rangle] + t_n^2[\langle y_1, v_{2,n}\rangle + \langle y_2, v_{1,n}\rangle + \langle v_1^1, v_2^1\rangle + t_n(\langle v_1^1, v_{2,n}\rangle + \langle v_{1,n}, v_2^1\rangle) + t_n^2\langle v_{1,n}, v_{2,n}\rangle]$.

Since

$$\langle y_1, v_{2,n}\rangle + \langle y_2, v_{1,n}\rangle + \langle v_1^1, v_2^1\rangle + t_n(\langle v_1^1, v_{2,n}\rangle + \langle v_{1,n}, v_2^1\rangle) + t_n^2\langle v_{1,n}, v_{2,n}\rangle$$

tends to $\langle y_1, v_2^2\rangle + \langle y_2, v_1^2\rangle + \langle v_1^1, v_2^1\rangle$, one has

$$\langle y_1, v_2^2\rangle + \langle y_2, v_1^2\rangle \in V^2(\langle F_1, F_2\rangle, x_0, z_0, \langle y_1, v_2^1\rangle + \langle y_2, v_1^1\rangle) - \langle v_1^1, v_2^1\rangle.$$

□

Remark 3.1. Set $M : X \times \mathbb{R} \to 2^{\mathbb{R}^{2m}}$ by $M(x,z) = \{(y_1,y_2) \in \mathbb{R}^{2m} \mid y_1 \in F_1(x),\ y_2 \in F_2(x)\ :\ \langle y_1, y_2\rangle = z\}$. Assertion (i) of Proposition 3.15 becomes

$$\mathrm{cl}[\bigcup_{(y_1,y_2)\in M(x_0,z_0)} \{\langle y_1, V^1(F_2,x_0,y_2)\rangle + \langle y_2, V^1(F_1,x_0,y_1)\rangle\}] \subseteq V^1(\langle F_1,F_2\rangle, x_0, z_0).$$

Since the outer product possesses clearly the same properties as those of the inner product: $(u+v)\diamond w = (u \diamond w) + (v \diamond w)$ and $(tu)\diamond v = t(u\diamond v)$ for $t \in \mathbb{R}$ (but instead of the commutative property we have $u \diamond w = (w \diamond u)^t$), we obtain the following rule (and a counterpart of Remark 3.1).

Proposition 3.16 (Outer Product Rule). *Let $x_0 \in \mathrm{dom}F_1 \cap \mathrm{intdom}F_2$ and $z_0 \in F_1 \diamond F_2(x_0)$ with $z_0 = y_1 \diamond y_2$ for $y_1 \in F_1(x_0)$ and $y_2 \in F_2(x_0)$.*

(i) *If F_2 has proto-variational set $V^1(F_2,x_0,y_2)$, then*

$$y_1 \diamond V^1(F_2,x_0,y_2) + V^1(F_1,x_0,y_1)\diamond y_2 \subseteq V^1(F_1\diamond F_2, x_0, z_0).$$

(ii) *If F_2 has proto-variational set $V^2(F_2,x_0,y_2,v_2^1)$, then*

$$y_1 \diamond V^2(F_2,x_0,y_2,v_2^1) + V^2(F_1,x_0,y_1,v_1^1)\diamond y_2$$

$$\subseteq V^2(F_1\diamond F_2, x_0, z_0, y_1\diamond v_2^1 + v_1^1 \diamond y_2) - v_1^1 \diamond v_2^1.$$

Proposition 3.17 (Quotient Rule). *Let $x_0 \in \mathrm{dom}F_1 \cap \mathrm{intdom}F_2$, $z_0 \in F_1/F_2$ and $z = y_1/y_2$ for $y_1 \in F_1(x_0)$ and $y_2 \in F_2(x_0)$. If F_2 has proto-variational set $V^1(F_2,x_0,y_2)$, then*

$$\frac{1}{y_2^2}(y_2V^1(F_1,x_0,y_1) - y_1V^1(F_2,x_0,y_2)) \subseteq V(F_1/F_2,x_0,z_0).$$

Proof. Let $v_1 \in V^1(F_1,x_0,y_1)$ and $v_2 \in V^1(F_2,x_0,y_2)$. For v_1 there exist $t_n \to 0^+, x_n \xrightarrow{F_1} x_0$ and $v_{1,n} \to v_1$ such that $y_1 + t_nv_{1,n} \in F_1(x_n)$ for all n. For v_2 and the above sequences t_n and x_n, there exists $v_{2,n} \to v_2$ such that $y_2 + t_nv_{2,n} \in F_2(x_n)$. Assume (by using a subsequence if necessary) that $y_2 + t_nv_{2,n} \neq 0$ for all n. Then,

$$\frac{y_1 + t_nv_{1,n}}{y_2+t_nv_{2,n}} = \frac{y_1}{y_2} + t_n\left[\frac{v_{1,n}y_2 - v_{2,n}y_1}{y_2^2 + t_nv_{2,n}y_2}\right]$$

belongs to $(F_1/F_2)(x_n)$ and

$$\frac{v_{1,n}y_2 - v_{2,n}y_1}{y_2^2 + t_n v_{2,n} y_2} \to \frac{y_2 v_1 - y_1 v_2}{y_2^2}.$$

Therefore,

$$\frac{1}{y_2^2}(y_2 v_1 - y_1 v_2) \in V^1(F_1/F_2, x_0, z_0).$$

$\square$

Corollary 3.18 (Reciprocal Rule). *Let $F : X \to 2^{\mathbb{R}}$ and $\frac{1}{z_0} \in F(x_0)$. Then*

$$-z_0^2 V^1(F, x_0, 1/z_0) \subseteq V^1(1/F, x_0, z_0).$$

Remark 3.2. If we define $M : X \times \mathbb{R} \to 2^{\mathbb{R}^2}$ by

$$M(x,z) := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in F_1(x), y_2 \in F_2(x) \ : \ y_1/y_2 = z\},$$

then the inclusion in Proposition 3.17 is equivalent to

$$\mathrm{cl}[\bigcup_{(y_1,y_2)\in M(x_0,z_0)} \frac{1}{y_2^2}(y_2 V^1(F_1, x_0, y_1) - y_1 V^1(F_2, x_0, y_2))] \subseteq V^1(F_1/F_2, x_0, z_0).$$

Proposition 3.19 (Maximum Rule). *Let $x_0 \in \mathrm{dom} F_1 \cap \mathrm{intdom} F_2$ and $z = \max\{y_1, y_2\}$ for $y_1 \in F_1(x_0)$ and $y_2 \in F_2(x_0)$. If F_2 has proto-variational set $V^1(F_2, x_0, y_2)$, then*

$$\alpha V^1(F_1, x_0, y_1) + \beta V^1(F_2, x_0, y_2) + \gamma(V^1(F_1, x_0, y_1) \vee V^1(F_2, x_0, y_2))$$
$$\subseteq V^1(F_1 \vee F_2, x_0, z_0),$$

where

$$\begin{cases} \alpha = 1, \beta = \gamma = 0, & \text{if } y_1 > y_2, \\ \beta = 1, \gamma = \alpha = 0, & \text{if } y_2 > y_1, \\ \gamma = 1, \alpha = \beta = 0, & \text{if } y_1 = y_2. \end{cases}$$

Proof. Let $v_1 \in V^1(F_1, x_0, y_1)$, $v_2 \in V^1(F_2, x_0, y_2)$ and $t_n \to 0^+$, $x_n \stackrel{F_1}{\to} x_0$, $v_{1,n} \to v_1$, $v_{2,n} \to v_2$ such that $y_1 + t_n v_{1,n} \in F_1(x_n)$ and $y_2 + t_n v_{2,n} \in F_2(x_n)$ for all n. Then,

$$\max\{y_1 + t_n v_{1,n}, y_2 + t_n v_{2,n}\} \in (F_1 \vee F_2)(x_n).$$

We rewrite the left-hand side as follows

$$\begin{aligned}
&\max\{y_1 + t_n v_{1,n}, y_2 + t_n v_{2,n}\} \\
&= \max\{y_1, y_2\} + \max\{y_1 + \min\{-y_1, -y_2\} + t_n v_{1,n}, y_2 + \min\{-y_1, -y_2\} + t_n v_{2,n}\} \\
&= \max\{y_1, y_2\} + t_n \max\{\min\{0, \frac{y_1 - y_2}{t_n}\} + v_{1,n}, \min\{\frac{y_2 - y_1}{t_n}, 0\} + v_{2,n}\} \\
&:= \max\{y_1, y_2\} + t_n w_n.
\end{aligned}$$

We have three cases. If $y_1 > y_2$, then

$$\min\{0, \frac{y_1 - y_2}{t_n}\} + v_{1,n} \to v_1,$$

$$\min\{0, \frac{y_2 - y_1}{t_n}\} + v_{2,n} \to -\infty.$$

Hence $w_n \to v_1$. Similarly, if $y_2 > y_1$, one has $w_n \to v_2$. If $y_1 = y_2$, then

$$\max\{\lim v_{1,n}, \lim v_{2,n}\} \to \max\{v_1, v_2\}.$$

By the definition of $V^1(F_1 \vee F_2, x_0, z_0)$ we are done. □

Similarly we have

Proposition 3.20 (Minimum Rule). *Let* $x_0 \in \mathrm{dom} F_1 \cap \mathrm{intdom} F_2$ *and* $z = \min\{y_1, y_2\}$ *for* $y_1 \in F_1(x_0)$ *and* $y_2 \in F_2(x_0)$. *If* F_2 *has proto-variational set* $V^1(F_2, x_0, y_2)$, *then*

$$\begin{aligned}
&\alpha V^1(F_1, x_0, y_1) + \beta V^1(F_2, x_0, y_2) + \gamma(V^1(F_1, x_0, y_1) \wedge V^1(F_2, x_0, y_2)) \\
&\subseteq V^1(F_1 \wedge F_2, x_0, z_0),
\end{aligned}$$

where

$$\begin{cases} \alpha = 1, \beta = \gamma = 0, & \text{if } y_1 < y_2, \\ \beta = 1, \gamma = \alpha = 0, & \text{if } y_2 < y_1, \\ \gamma = 1, \alpha = \beta = 0, & \text{if } y_1 = y_2. \end{cases}$$

Remark 3.3. Define $M_1, M_2 : X \times \mathbb{R} \to 2^{\mathbb{R}^2}$ by

$$M_1(x, z) := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in F_1(x), y_2 \in F_2(x) \ : \ \max\{y_1, y_2\} = z\},$$

$$M_2(x, z) := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in F_1(x), y_2 \in F_2(x) \ : \ \min\{y_1, y_2\} = z\}.$$

Then, the inclusions asserted in Propositions 3.19 and 3.20 are rewritten equivalently as follows

(i) $\text{cl}[\bigcup_{(y_1,y_2)\in M_1(x_0,z_0)} \{\alpha V^1(F_1, x_0, y_1) + \beta V^1(F_2, x_0, y_2) + \gamma(V^1(F_1, x_0, y_1) \vee V^1(F_2, x_0, y_2))\}]$
$\subseteq V^1(F_1 \vee F_2, x_0, z_0)$;

(ii) $\text{cl}[\bigcup_{(y_1,y_2)\in M_2(x_0,z_0)} \{\alpha V^1(F_1, x_0, y_1) + \beta V^1(F_2, x_0, y_2) + \gamma(V^1(F_1, x_0, y_1) \wedge V^1(F_2, x_0, y_2))\}]$
$\subseteq V^1(F_1 \wedge F_2, x_0, z_0)$.

4. Applications: optimality conditions in nonsmooth vector optimization

Unlike the scalar case, in vector optimization there are a variety of concepts of solutions; all of them are significant to extents. Pareto and weak efficient solutions have been most investigated in the literature. Recently, we also contributed to considerations of ideal and firm (called also strict) solutions [9-12]. A common observation is that ideal solutions are too rare and sets of weak and Pareto solutions are rather large and some of these solutions may have abnormal properties. Hence, a number of notions of proper solutions (known also as proper efficiency) have been playing important roles. For treatments and comparisons of various proper efficiencies see e.g. [6, 13, 14, 15, 16]. In [16] the definition of

D-efficiency is proposed to include many kinds of known proper efficiencies, with D being a family of the so-called dilating cones. Very recently, [6] introduced Q-minimality notion to contain not only more kinds of proper efficiency but also the weak and ideal solutions. In this section we discuss optimality conditions for the Benson properness, as an example of the known kinds of proper efficiency, and the Q-minimality, as a very general optimality notion. Let X, Y and Z be a normed spaces; $C \subseteq Y$ and $D \subseteq Z$ closed, pointed convex cones with nonempty interior; and $F : X \to 2^Y$, $G : X \to 2^Z$. Our vector optimization problem is

$$\text{(P)} \qquad \min F(x), \text{ s.t. } G(x) \cap -D \neq \emptyset.$$

Set $A := \{x \in X : G(x) \cap -D \neq \emptyset\}$ (the feasible set) and $F(A) := \bigcup\limits_{x \in A} F(x)$. Recall that [5], for $x_0 \in A$ and $y_0 \in F(x_0)$, (x_0, y_0) is called local Benson-proper solution (or local Benson-properly efficient pair) of (P) if there exists $U \in U(x_0)$ such that

$$\text{clcone}(F(U \cap A) + C - y_0) \cap -C = \{0\}.$$

Let $Q \subseteq Y$ be an arbitrary nonempty open cone (not necessarily convex) different from Y. We say that (x_0, y_0) is a local Q-minimal solution (or local Q-minimal pair) of (P), see [6], if there exists $U \in U(x_0)$ such that

$$(F(U \cap A) - y_0) \cap -Q = \emptyset.$$

Since Q is not required to be convex, Q-minimality includes additionally many notions of efficiency such as the ideal efficiency, the Hurwicz and Benson proper efficiencies, see [6]. The following fact is often used in this section.

Lemma 4.1. *Let $Q \subseteq X$ be an open cone, not necessarily convex, $x_0 \in \text{bd}Q$, $x \in \text{intcone}(Q - x_0)$, $s_n \to 0^+$ and $\frac{1}{s_n}(x_n - x_0) \to x$. Then $x_n \in Q$ for large n.*

Proof. Take an open neighborhood U of x, contained in cone$(Q - x_0)$, of the form $\{\lambda(q - x_0)|\ q \in Q_1, \lambda \in (\lambda_1, \lambda_2)\}$, where $Q_1 \subseteq Q$ is open and bounded and $\lambda_1, \lambda_2 > 0$. Then, $\text{cone}_+U = \{\lambda(q - x_0)|\lambda > 0, q \in Q_1\} \subseteq cone(Q - x_0)$.

Suppose there is a subsequence, denoted also by $\{x_n\}$, with $x_n \notin Q$ for all n . Then $x_n - x_0 \notin \text{cone}_+U$ for all n. On the other hand, we must have $\frac{1}{s_n}(x_n - x_0) \in U$ and then $x_n - x_0 \in \text{cone}_+U$, for all n, a contradiction. □

4.1. Optimality conditions for Benson-proper efficiency

Theorem 4.1. *Let (x_0, y_0) be a local Benson-proper solution of problem* (P) *and $z_0 \in G(x_0) \cap -D$. Assume that either C has a weakly compact base and $F_+(A)$ is convex or C has a compact base. Then the following assertions hold.*

(i) *There exists a closed convex pointed cone S such that $C \setminus \{0\} \subseteq \text{int}S$ and*

$$\text{clcone}(F(U \cap A) + C - y_0) \cap -\text{int}S = \emptyset.$$

(ii) *The following separations hold*

(ii$_1$) *$V^1((F, G)_+, x_0, (y_0, z_0)) \cap -\text{int}(S \times D(z_0)) = \emptyset$;*

(ii$_2$) *if $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \cap -\text{bd}(S \times D(z_0))$,*

$(u_2, v_2) \in V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \cap -\text{bd}(S(u_1) \times D(z_0)), ...,$

$(u_{m-1}, v_{m-1}) \in V^{m-1}((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2}))$

$\cap - \text{bd}(S(u_1) \times D(z_0)), m \geq 2$, then

$$V^m((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \cap -\text{int}(S(u_1) \times D(z_0)) = \emptyset. \quad (1)$$

Proof. (i) See [17].

(ii) If $(u_1, v_1) = ... = (u_{m-1}, v_{m-1}) = (0, 0)$, assertion (ii$_2$) collapses to (i$_1$). Hence, it suffices to demonstrate (ii$_2$). Suppose there exists (y, z) in the left-hand side of (1). Then, there are $x_n \overset{(F,G)}{\to} x_0,\ t_n \to 0^+$ and $(y_n, z_n) \in (F,G)(x_n) + C \times D$ such that

$$\frac{1}{t_n^2}\big((y_n, z_n) - (y_0, z_0) - t_n(u_1, v_1) - ... - t_n^{m-1}(u_{m-1}, v_{m-1})\big) \to (y, z),$$

Consequently, one has $\alpha_i \geq 0$ and $h_i \in S$ such that $u_i = -\alpha_i(h_i + u_1)$ and

$$\frac{1}{t_n^m}(y_n - y_0 - t_n u_1 - ... - t_n^{m-1}u_{m-1}) = \frac{1}{t_n^m}(y_n - y_0 - t_n u_1 + \sum_{i=2}^{m-1} \alpha_i t_n^i (h_i + u_1))$$

$$= \left(\frac{y_n - y_0 + \sum\limits_{i=2}^{m-1} \alpha_i t_n^i h_i}{t_n(1 - \sum\limits_{i=2}^{m-1} \alpha_i t_n^{i-1})} - u_1 \right) \frac{1 - \sum\limits_{i=2}^{m-1} \alpha_i t_n^{i-1}}{t_n^{m-1}} \to y.$$

By virtue of Lemma 4.1, for n large enough, we have

$$y_n - y_0 + \sum_{i=2}^{m-1} \alpha_i t_n^i h_i \in -\text{int}S$$

and hence

$$y_n - y_0 \in -\text{int}S.$$

Similarly, for $i = 1, ..., m-1$ as $v_i \in -\text{cone}(D + z_0)$ there are $\beta_i \geq 0$ and $d_i \in D$ with $v_i = -\beta_i(d_i + z_0)$. Therefore,

$$\frac{1}{t_n^m}(z_n - z_0 - t_n v_1 - ... - t_n^{m-1}v_{m-1}) = \frac{1}{t_n^m}(z_n - z_0 - \sum_{i=1}^{m-1} \beta_i t_n^i (d_i + z_0))$$

$$= \left(\frac{z_n + \sum\limits_{i=1}^{m-1} \beta_i t_n^i d_i}{1 - \sum\limits_{i=1}^{m-1} \beta_i t_n^i} - z_0 \right) \frac{1 - \sum\limits_{i=1}^{m-1} \beta_i t_n^i)}{t_n^m} \to z.$$

Using again Lemma 4.1 yields $z_n \in -\text{int}D$. On the other hand, there exist $(\overline{y_n}, \overline{z_n}) \in (F,G)(x_n)$ and $(\overline{c_n}, \overline{d_n}) \in C \times D$ such that

$$(y_n, z_n) = (\overline{y_n}, \overline{z_n}) + (\overline{c_n}, \overline{d_n}).$$

Hence, for sufficiently large n that $\overline{y_n} + \overline{c_n} - y_0 \in -\text{int}S$ and $\overline{z_n} + \overline{d_n} \in -\text{int}D$ contradicting (i). □

Similarly one has the corresponding result using W^m as follows.

Theorem 4.2. *Assume that (x_0, y_0) is a local Benson-properly efficient pair of problem* (P)*, $z_0 \in G(x_0) \cap -D$ and either C has a weakly compact base and $F_+(A)$ is convex or C has a compact base. Then the following assertions hold.*

(i) *There exists a closed convex pointed cone S such that $C \setminus \{0\} \subseteq \mathrm{int}S$ and*

$$\mathrm{clcone}(F(U \cap A) + C - y_0) \cap -\mathrm{int}S = \emptyset.$$

(ii) *The following separations hold*

(ii$_1$) *$W^1((F, G)_+, x_0, (y_0, z_0)) \cap -\mathrm{int}(S \times D) = \emptyset$;*

(ii$_2$) *if $(u_1, v_1) \in W^1((F, G)_+, x_0, (y_0, z_0)) \cap -\mathrm{bd}(S \times D)$,*

$(u_2, v_2) \in W^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \cap -\mathrm{bd}(S(u_1) \times D(v_1)), ...,$

$(u_{m-1}, v_{m-1}) \in W^{m-1}((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \cap -\mathrm{bd}(S(u_1) \times D(v_1)), m \geq 2$, *then*

$$W^m((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \cap -\mathrm{int}(S(u_1) \times D(v_1)) = \emptyset.$$

As far as we know there have not been higher-order optimality conditions for Benson proper efficiency in the literature. Now we pass to sufficient optimality conditions. We need the following generalized convexity, which is motivated by a more restrictive condition defined in [18].

Definition 4.1

(i) $F : X \to 2^Y$ is called $C^\sharp$-variational pseudoconvex at $(x_0, y_0) \in \mathrm{gr}F$ if, there exists $c^* \in C^\sharp$ such that from $c^*(F(x) - y_0) \cap (-\infty, 0) \neq \emptyset$ for some $x \in X$ one has $c^*(V^1(F, x_0, y_0)) \cap (-\infty, 0) \neq \emptyset$.

(ii) $F : X \to 2^Y$ is called C^*-variational pseudoconvex at $(x_0, y_0) \in \text{gr}F$ if, there is $c^* \in C^* \setminus \{0\}$ such that from $c^*(F(x) - y_0) \cap (-\infty, 0] \neq \emptyset$ for some $x \in X$ it follows that $c^*(V^1(F, x_0, y_0)) \cap (-\infty, 0] \neq \emptyset$.

Theorem 4.3. *Let* $F : X \to 2^Y$, $G : X \to 2^Z$ *and* $(x_0, y_0) \in \text{gr}F$. *Assume that*

(i) *F is a $C^\sharp$-variational pseudoconvex at (x_0, y_0) and G is D^*-variational pseudoconvex at (x_0, z_0) for some $z_0 \in G(x_0) \cap -D$;*

(ii) *For each $z_0 \in G(x_0) \cap -D$, there exist $c^* \in C^\sharp$ and $d^* \in D^*$ such that*

$$\inf[c^*(V^1(F, x_0, y_0)) + d^*(V^1(G, x_0, z_0)] \geq 0,$$

$$d^*(G(x_0) \cap -D) = \{0\}.$$

Then (x_0, y_0) is a Benson-properly efficient solution of (P).

Proof. Suppose, ad absurdum, there exists a nonzero point $y \in \text{clcone}(F(A) + C - y_0) \cap -C$. Then $c^*(y) < 0$ and there exist positive λ_n, $x_n \in A$, $y_n \in F(x_n)$ and $c_n \in C$ such that

$$c^*(y) = \lim_{n \to \infty} \lambda_n (c^*(y_n - y_0) + c^*(c_n)).$$

Hence $\lim\limits_{n \to \infty} \lambda_n c^*(y_n - y_0) < 0$. Then $c^*(y_n - y_0) < 0$, for large n. By the assumed pseudoconvexity of F,

$$c^*(V^1(F, x_0, y_0)) \cap (-\infty, 0) \neq \emptyset.$$

Since $x_n \in A$, there exists $z_n \in G(x_n) \cap -D$. By assumption (ii) $d^*(z_n - z) \leq 0$ for any $z \in G(x_0) \cap -D$. By the pseudoconvexity of G, $d^*(V^1(G, x_0, z_0)) \cap (-R_+) \neq \emptyset$. Hence

$$[c^*(V^1(F, x_0, y_0)) + d^*(V(G, x_0, z_0)] \cap (-\infty, 0) \neq \emptyset,$$

which is a contradiction. □

4.2. Optimality conditions for Q-minimal solutions

Theorem 4.4. *Assume that (x_0, y_0) is a local Q-minimal solution of problem* (P) *and $z_0 \in G(x_0) \cap -D$. Then*

(i) $V^1((F,G), x_0, (y_0, z_0)) \bigcap -(Q \times \mathrm{int}D(z_0)) = \emptyset$;

(ii) *if $(u_1, v_1) \in V^1((F,G), x_0, (y_0, z_0)) \bigcap -\mathrm{bd}(Q \times D(z_0))$, then*

$$V^2((F,G), x_0, (y_0, z_0), (u_1, v_1) \bigcap -\mathrm{int}(Q(u_1) \times D(z_0)) = \emptyset;$$

(iii) *if Q is additionally convex and $(u_1, v_1) \in V^1((F,G), x_0, (y_0, z_0)) \bigcap -\mathrm{bd}(Q \times D(z_0))$, $(u_2, v_2) \in V^2((F,G), x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\mathrm{bd}(Q(u_1) \times D(z_0)), \ldots,$*

$(u_{m-1}, v_{m-1}) \in V^{m-1}((F,G), x_0, (y_0, z_0), (u_1, v_1), \ldots, (u_{m-2}, v_{m-2}))$

$\bigcap -\mathrm{bd}(Q(u_1) \times D(z_0)), m \geq 2$, *then*

$$V^m((F,G), x_0, (y_0, z_0), (u_1, v_1), \ldots, (u_{m-1}, v_{m-1}))$$
$$\bigcap -\mathrm{int}(Q(u_1) \times D(z_0)) = \emptyset.$$

Proof. (i) and (ii) If $(u_1, v_1) = (0, 0)$, assertion (ii) collapses to (i). Hence, it suffices to prove (ii). Suppose to the contrary, there exists (y, z) in the intersection needed to be shown empty. There are then $x_n \overset{(F,G)}{\rightarrow} x_0$, $t_n \to 0^+$ and $(y_n, z_n) \in (F,G)(x_n)$ such that

$$\frac{1}{t_n^m}\Big((y_n, z_n) - (y_0, z_0) - t_n(u_1, v_1)\Big) \to (y, z),$$

where $y \in -\mathrm{int}Q(u_1)$ and $z \in -\mathrm{int}D(z_0)$. Then,

$$\frac{1}{t_n}\Big(\frac{1}{t_n}(y_n - y_0) - u_1\Big) \to y$$

and Lemma 4.1 gives $y_n - y_0 \in -Q$ for large n. Similarly, this lemma asserts that $z_n - t_n v_1 \in -\mathrm{int}D$ for large n, and hence $z_n \in -\mathrm{int}D$. This contradicts the local Q-minimality of (x_0, y_0).

(iii) Arguing also by contraposition, this time we have similar sequences x_n, t_n and (y_n, z_n) such that

$$= \left(\frac{y_n - y_0 + \sum_{i=2}^{m-1} \alpha_i t_n^i q_i}{t_n(1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1})} - u_1 \right) \frac{1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1}}{t_n^{m-1}} \to y.$$

As $s_n = t_n^{m-1}(1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1})^{-1} \to 0^+$, for large n we have, by Lemma 4.1,

$$y_n - y_0 + \sum_{i=2}^{m-1} \alpha_i t_n^i q_i \in -Q,$$

and then (as Q is convex) $y_n - y_0 \in -Q$.

Similarly, for $i = 1, ..., m-1$, there are $\beta_i \geq 0$ and $d_i \in D$ such that $v_i = -\beta_i(d_i + z_0)$. Therefore,

$$\frac{1}{t_n^m}(z_n - z_0 - t_n v_1 - ... - t_n^{m-1} v_{m-1})$$
$$= \left(\frac{z_n + \sum_{i=1}^{m-1} \beta_i t_n^i d_i}{1 - \sum_{i=1}^{m-1} \beta_i t_n^i} - z_0 \right) \frac{1 - \sum_{i=1}^{m-1} \beta_i t_n^i}{t_n^m} \to z.$$

Again Lemma 4.1 yields that $z_n \in -\text{int } D$. So, we have arrived at a contradiction. □

Similarly, we have the following necessary condition using the variational set of type 2.

Theorem 4.5. *Assume the same as for Theorem 4.4. Then*

(i) $W^1((F,G), x_0, (y_0, z_0)) \bigcap -(Q \times \text{int} D) = \emptyset$;

(ii) *if* $(u_1, v_1) \in W^1((F,G), x_0, (y_0, z_0)) \bigcap -\text{bd}(Q \times D)$, *then*

$$W^2((F,G), x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{int}(Q(u_1) \times D(v_1)) = \emptyset;$$

(iii) *if* Q *is additionally convex and* $(u_1, v_1) \in W^1((F,G), x_0, (y_0, z_0)) \bigcap -\text{bd}(Q \times D)$,

$(u_2,v_2) \in W^2((F,G),x_0,(y_0,z_0),(u_1,v_1)) \bigcap -\mathrm{bd}(Q(u_1)\times D(v_1)), ...,$

$(u_{m-1},v_{m-1}) \in W^{m-1}((F,G),x_0,(y_0,z_0),(u_1,v_1),...,(u_{m-2},v_{m-2}))$

$\bigcap -\mathrm{bd}(Q(u_1)\times D(v_1)), m \geq 2,$ *then*

$$W^m((F,G),x_0,(y_0,z_0),(u_1,v_1),...,(u_{m-1},v_{m-1})) \bigcap -\mathrm{int}(Q(u_1)\times D(v_1)) = \emptyset.$$

Remark 4.1. The assumed convexity of Q in Theorems 4.4 (iii) and 4.5 (iii) does not restrict much the generality, since in this case a Q-minimal solution still encompasses the following solutions (see [6], Theorem 21.7): weak efficient, positive proper, Henig- and strong Henig-proper, and (supposing $\mathrm{int}C^*$ is nonempty) supper efficient, with Q being suitably chosen for each case. By Theorem 4.1 (i) a Benson-proper solution is a Q-minimal solution as well. We skip a recalling the definitions of these kinds of solutions here; the interested reader is referred to [6, 14, 16].

With relaxed convexity assumptions we establish the following sufficient condition, including stronger separations (with $(F,G)_+$). Remember that here Q is not necessarily convex.

Theorem 4.6. *For problem* (P)*, let* $x_0 \in A, y_0 \in F(x_0)$ *and* $z_0 \in G(x_0) \cap -D$*. Assume that either at* x_0*,* F *is* C*-star-shaped and* G *is* D*-star-shaped or* (F,G) *is pseudoconvex at* $(x_0,(y_0,z_0))$*. Then* (x_0,y_0) *is a (global)* Q*-minimal solution if either of the following is satisfied*

(i) $V^1((F,G)_+,x_0,(y_0,z_0)) \bigcap -(Q\times D(z_0)) = \emptyset$;

(ii) if $(u_1,v_1) \in V^1((F,G)_+,x_0,(y_0,z_0)) \bigcap -\mathrm{bd}(Q\times D(z_0)), (u_2,v_2) \in$
$V^2((F,G)_+,x_0,(y_0,z_0),(u_1,v_1)) \bigcap -\mathrm{bd}(Q(u_1)\times D(z_0)), ..., (u_{m-1},v_{m-1}) \in$

$V^{m-1}((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd}(Q(u_1) \times D(z_0))$, $m \geq 2$, *then*

$$V^m((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-1}, v_{m-1}))$$
$$\bigcap -(Q(u_1) \times D(z_0)) = \emptyset.$$

Proof. If $(u_1,v_1) = ... = (u_{m-1}, v_{m-1}) = (0,0)$, (ii) becomes (i). Therefore, we need to prove only that the conclusion holds under condition (i). By Proposition 2.4, one obtains

$$((F,G)(x) - (y_0,z_0)) \bigcap -(Q \times D(z_0)) = \emptyset.$$

If one had $x \in A$ and $y \in F(x)$ such that $y - y_0 \in -Q$. Then there was $z \in G(x) \cap -D$ satisfying

$$(y,z) - (y_0,z_0) \in -(Q \times D(z_0)),$$

a contradiction. □

To the best of our knowledge the preceding results are the first contribution to higher-order optimality conditions for Q-minimality.

References

[1] P.Q. Khanh and N.D. Tuan, Variational sets of multivalued mappings and a unified study of optimality conditions, J. Optim. Theory Appl. 139 (2008) 45-67.

[2] P. Q. Khanh and N.D. Tuan, Higher-order variational sets and higher-order optimality conditions for proper efficiency in set-valued nonsmooth vector optimization, J. Optim. Theory Appl. 139 (2008) 243-261.

[3] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol. I-Basic Theory, Springer, Berlin, 2006.

[4] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Vol. II-Applications. Springer, Berlin, 2006.

[5] H.P. Benson, An improved definition of proper efficiency for vector minimization with respect to cones, J. Math. Anal. Appl. 71 (1978) 232-241.

[6] T.X.D. Ha, Optimality conditions for several types of efficient solutions of set-valued optimization problems, Chap. 21 in Nonlinear Analysis and Variational Problems in: P. Pardalos, Th. M. Rassis and A. A. Khan (Eds), 2009, pp. 305-324.

[7] R.T. Rockafellar, R.J.B. Wets, Variational Analysis, Springer, Berlin, 1998.

[8] B.S. Mordukhovich, Generalized differential calculus for nonsmooth and set-valued mappings, J. Math. Anal. Appl. 183 (1994) 250-288.

[9] P.Q. Khanh, N.D. Tuan, First and second order optimality conditions using approximations for nonsmooth vector optimization in Banach spaces. J. Optim. Theory Appl. 130 (2006) 289-308.

[10] Khanh, P. Q.; Tuan, N. D. Optimality conditions for nonsmooth multiobjective optimization using Hadamard directional derivatives. J. Optim. Theory Appl. 133 (2007) 341-357.

[11] P.Q. Khanh, Tuan, N.D. Tuan, First and second-order approximations as derivatives of mappings in optimality conditions for nonsmooth vector optimization. Appl. Math. Optim. 58 (2008) 147-166.

[12] P.Q. Khanh, N.D. Tuan, Optimality conditions using approximations for nonsmooth vector optimization problems under general inequality constraints. J. Convex Anal. 16 (2009) 169 -186.

[13] A. Guerraggio , E. Molho and A. Zaffaroni, On the notion of proper efficiency in vector optimization, J. Optim. Theory Appl. 82 (1994) 121.

[14] P.Q. Khanh, Proper solutions of vector optimization problems. J. Optim. Theory Appl. 74 (1992) 105-130.

[15] P.Q. Khanh, Optimality conditions via norm scalarization in vector optimization, SIAM J. Control Optim. 31 (1993) 646-658.

[16] E.K. Makarov, N.N. Rachkovski, Unified representation of proper efficiency by means of dilating cones, J. Optim. Theory Appl. 101(1999) 141-165.

[17] Dauer, J. P. and Saleh, J. P. , A characterization of proper minimal problems as a solution of sublinear optimization problems, J. Math. Anal. Appl. 178(1993) 227-246.

[18] B.H. Sheng, S.Y. Liu, On the generalized Fritz John optimality conditions of vector optimization with set-valued maps under Benson proper efficiency, Appl. Math. Mech. (English Ed.), 23 (2002) 71-78.